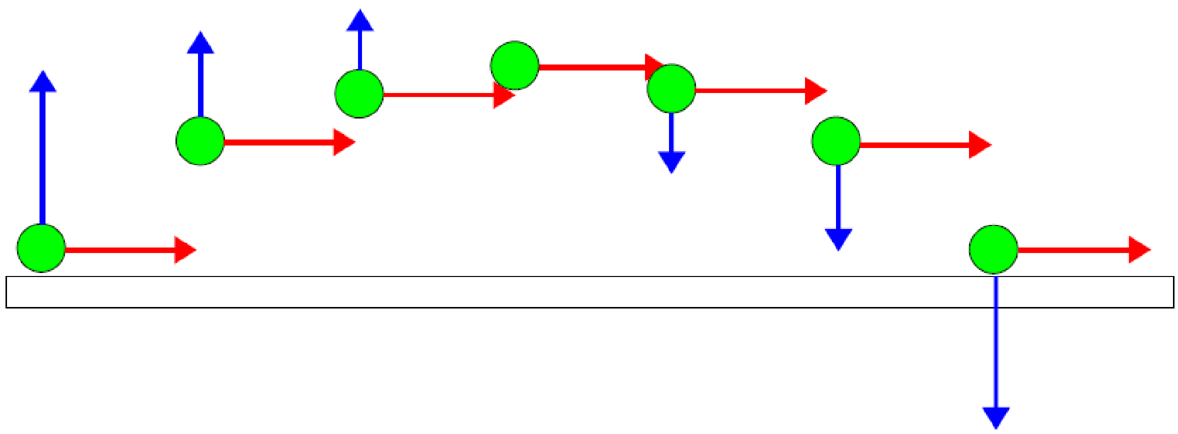
Projectile Motion at an Angle

To do questions involving objects launched upwards at an angle we need to add a few more steps to our problem solving from the previous lesson.

● There are actually two ways to do these types of problems, one based on the vertical velocity of the object, the other based on the vertical displacement.
● The only big difference between these methods is how we are going to calculate the time the object spends in the air.
● It's up to you to decide which method you are most comfortable with, and which method suits the particular question you are doing. Below we will look at both methods, and then it's up to you to figure out how you will approach each problem.

Imagine for a moment that you are watching a soccer ball as it rises into the air after you kick it upwards at an angle. Look at the diagram below as you read through the following description.



● When the ball leaves your foot it is going at the fastest speed that it can possibly move during its flight.
● The instant it leaves your foot, gravity is pulling down on it, causing it to have less and less vertical velocity.
● Remember that there will be no change in the horizontal component of its velocity throughout the entire flight since there is nothing to slow it down in the horizontal direction (the red vector never changes).
● When it reaches the highest point in its flight, it isn't moving up and it isn't moving down for an instant in time… its vertical velocity is ZERO!
● By the time it reaches the ground again, it will still be moving with its original horizontal velocity and will have just as much vertical velocity as it had when it left your foot. However, while the horizontal velocity will be the same, the vertical velocity will now be in the opposite direction! We can use this information about its vertical movement to do some calculations.

We know...
● that there is gravity ($-9.80 m/s^2$) causing the acceleration on the object vertically.
● the initial vertical velocity of the object.
● the final vertical velocity of the object (since it's just the negative of the initial vertical velocity).

If you're feeling a bit confused at this point, just relax, once we do some examples it will become a lot more clear ☺

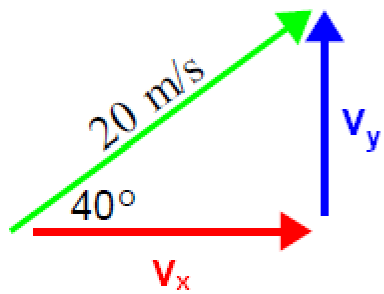
Important Note!!!

From now on, unless stated otherwise, it will be assumed that we are using the coordinate system where up and right are positive. This goes for every example in Lesson 6 and all proceeding lessons (i.e. Lesson 7, 8, 9, 10, etc.).

Example 1

You kick a soccer ball at an angle of 40° above the ground with a velocity of 20m/s. Determine:
a) How high it will go.
b) How much time it spends in the air.
c) How far away from you it will hit the ground (aka the range).
d) What the ball's velocity will be when it hits the ground.

Before we can calculate anything else, we first need to break the original velocity into components, v_x and v_y.



$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{adj}{hyp}$$

$$\cos\theta = \frac{x}{hyp}$$

$$\cos\theta = \frac{v_x}{v_{total}}$$

$$(v_{total})(\cos\theta) = v_x$$

$$v_x = (v_{total})(\cos\theta) = (20m/s)(\cos 40°) = 15m/s$$

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{opp}{hyp}$$

$$\sin\theta = \frac{y}{hyp}$$

$$\sin\theta = \frac{v_y}{v_{total}}$$

$$(v_{total})(\sin\theta) = v_y$$

$$v_y = (v_{total})(\sin\theta) = (20m/s)(\sin 40°) = 13m/s$$

a) “Think vertical” – How high will it go?
At its maximum height, halfway through its flight, the object won't be going up or down, so we'll say that its final velocity at that point is zero.

Writing down the information we have:

v_{fy} = 0m/s
v_{iy} = 13m/s
a_y = a = g = -9.80m/s^2
d_y = ?

$$v_{fy}{}^2 = v_{iy}{}^2 + 2a_y d_y$$
$$v_{fy}{}^2 - v_{iy}{}^2 = 2a_y d_y$$
$$\frac{v_{fy}{}^2 - v_{iy}{}^2}{2a_y} = d_y$$
$$d_y = \frac{v_{fy}{}^2 - v_{iy}{}^2}{2a_y} = \frac{(0m/s)^2 - (13m/s)^2}{2(-9.80m/s^2)} = 8.6m$$

The ball will reach a maximum height of 8.6 m.

b) “Think vertical” – How much time will it spend in the air?
The following three methods show slightly different ways of approaching the problem.

● In the first, we will assume that the ball strikes the ground at the same speed it had when it was kicked, but traveling in the opposite direction.
v_{iy} = 13m/s
v_{fy} = -13m/s
a_y = -9.80m/s^2
t = ?

● In the second, we will calculate the time it takes for the ball to reach the halfway point. Note that at the halfway point the vertical velocity is zero and the ball will have been in the air for exactly half of its total flight time. Thus, if we simply double our answer we will get the full time the ball is in the air.
v_{iy} = 13m/s
v_{fy} = 0m/s
a_y = -9.80m/s^2
t = ?

● In the third method we will assume that the ball strikes the ground at the same height that it left the ground from. This means that it has a vertical displacement of zero.

v_{iy} = 13m/s
d_y = 0m
a_y = -9.80m/s^2
t = ?

Method 1	***Method 2***	***Method 3***
$a_y = \frac{v_{fy} - v_{iy}}{t}$ $t = \frac{v_{fy} - v_{iy}}{a_y}$ $t = \frac{(-13m/s) - (13m/s)}{-9.80m/s^2}$ $t = 2.7s$	$a_y = \frac{v_{fy} - v_{iy}}{t}$ $t = \frac{v_{fy} - v_{iy}}{a_y}$ $t = \frac{(0m/s) - (13m/s)}{-9.80m/s^2}$ $t = 1.32653s$ ***Multiply by 2 and we find:*** $2 \times 1.32653s = 2.65306s = 2.7s$	$d_y = v_{iy}t + \frac{1}{2}a_y t^2$ $0m = v_{iy}t + \frac{1}{2}a_y t^2$ $-v_{iy}t = \frac{1}{2}a_y t^2$ $-v_{iy} = \frac{1}{2}a_y t$ $-2v_{iy} = a_y t$ $\frac{-2v_{iy}}{a_y} = t$ $t = \frac{-2v_{iy}}{a_y}$ $t = \frac{-2(13m/s)}{(-9.80m/s^2)} = 2.7s$

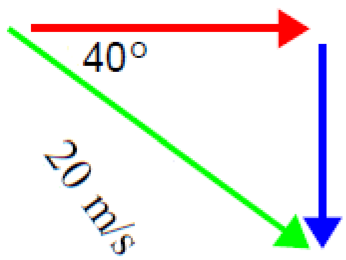
c) "Think horizontal" – How far away from you will it hit the ground (aka the range)?
It is moving at a constant velocity horizontally during the whole time we just figured out, so let's use the horizontal component of the velocity to figure out the displacement horizontally.

v_x = 15m/s
t = 2.7s
d_x = ?

$$v_x = \frac{d_x}{t}$$
$$v_x t = d_x$$
$$d_x = v_x t = (15m/s)(2.7s) = 41m$$

d) What is the ball's velocity when it hits the ground?

The ball's velocity when it hits the ground is exactly the same as when it was originally launched… 20m/s at 40° above the horizontal. The only difference is that now it's aimed into the ground. The answer then is 20m/s at 40° below the horizontal.



If you do not like this answer, you could also answer in component form.

- Since the horizontal velocity never changes, $v_x = v_{fx} = 15m/s$
- Since the vertical velocity will be the opposite of the initial vertical velocity, $v_{fy} = -13m/s$